J. Fluid Mech. (2009), *vol.* 634, *pp.* 499–507. © 2009 Cambridge University Press
doi:10.1017/S0022112009990759 Printed in the United Kingdom

Formation of singularities in Hall magnetohydrodynamics

MANUEL NÚÑEZ†
Departamento de Análisis Matemático, Universidad de Valladolid, 47005 Valladolid, Spain

(Received 30 April 2009 and in revised form 8 June 2009)

Hall magnetohydrodynamics has proved a useful model in several physical phenomena, in particular, in fast magnetic reconnection. The induction equation for this model involves the nonlinear Hall term which has been suspected to imply loss of regularity and in particular formation of singularities of the current density. Numerical simulations strongly suggest that this is the case, but so far a rigorous proof was lacking. We show that for a particular axisymmetric geometry, certain integrals of the magnetic field satisfy a differential inequality that leads to a finite-time blow-up of the field gradient, and therefore of the current density. We may interpret this as a breakdown of the field regularity and the formation of discontinuous solutions.

1. Introduction

Among the cascade of simplifications which start from the Boltzmann equations for multi-species plasmas and eventually end in the system of classical magnetohydrodynamics (MHD) (Chen 1983), the system of Hall MHD remains one of the most useful. It has been proposed to explain fast magnetic reconnection (Shay *et al.* 2001; Biskamp 2000) in order to avoid the shortcomings of the single fluid Sweet–Parker model (Sonnerup 1970; Yeh & Axford 1970). The problem of this is that in weakly collisional plasmas where the Spitzer resistivity is small, very thin current sheets are needed for resistive dissipation to be relevant. The thinness of the current sheet impedes the flow through it, which slows the reconnection process. In Hall MHD the relaxation of field lines appears to be governed by the whistler wave, which is dispersive and does not throttle mass flow.

Let us write the system: for recent explanations of this model and its range of validity (Pandey & Wardle 2008; Schekochihin *et al.* 2009), let $\mathbf{v}$ represent the ion velocity, $\mathbf{B}$ the magnetic field, $\mathbf{J}=\nabla\times\mathbf{B}$ the current density, P the kinetic pressure and d_i the ion skin depth. This quantity is defined as follows: let m_i be the ions mass, Z their charge, n_i the number density and $-e$ the electron charge. The ion plasma frequency is

$$\omega_{pi}=\left(\frac{4\pi n_i Z^2 e^2}{m_i}\right)^{1/2}.$$

Then $d_i=c/\omega_{pi}$. If we normalize the system by taking the density equal to one, as well as the speed of light, the equations of incompressible inviscid perfectly conducting

† Email address for correspondence: mnjmhd@am.uva.es

Hall MHD are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \mathbf{J}\times\mathbf{B} - \nabla P, \tag{1.1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) - d_i\nabla\times(\mathbf{J}\times\mathbf{B}), \tag{1.2}$$

$$\nabla\cdot\mathbf{v} = 0. \tag{1.3}$$

It is apparent that the right-hand side of the induction equation (1.2) presents the second-order Hall term $\nabla\times(\mathbf{J}\times\mathbf{B})$, which is nonlinear on the field and does not bode well for regularity results. Nonetheless, if we assume, for example, that $\mathbf{v}\cdot\mathbf{n}$ vanishes at the boundary of the domain Ω under consideration, and

$$\int_{\partial\Omega}(\mathbf{J}\times\mathbf{B})\cdot(\mathbf{B}\times\mathbf{n})\,\mathrm{d}V = 0, \tag{1.4}$$

then the total energy is conserved in time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}(v^2+B^2)\,\mathrm{d}V = 0. \tag{1.5}$$

This also occurs for $\mathbf{v}$ and $\mathbf{B}$ periodic in a periodic box, or when we have some combination of these conditions in different parts of $\partial\Omega$. Hence it is pointless to look for a blow-up of the solutions in the sense that one of them may grow indefinitely in L^2 norm in a finite time. What is possible is loss of regularity: solutions becoming discontinuous, for example, with the formation of shocks. That would be interesting from a mathematical viewpoint: most fluid equations possess theorems of existence and regularity for a finite time, but the most determined efforts have failed to provide either global theorems or examples of blow-up in a finite time. This is true not only in the celebrated instance of the Navier–Stokes equations but also for the Euler equations, for which less regularity should be expected (see, e.g. Deng, Hou & Yu 2006 and the references therein). Numerical studies suggest that rapid loss of smoothness is indeed possible (Kerr 1993; Peltz 2001). For the unphysical case of infinite mass or energy, blow-ups have been proved to exist (Li & Wang 2006; Gibbon, Moore & Stuart 2003). As for the Hall MHD system, proof of shock formation would be interesting not only mathematically but also physically, since this would allow the possibility of spontaneous formation of current sheets starting from smooth initial conditions: current sheets are the favoured configuration for magnetic reconnection studies. In fact, numerical modelizations strongly indicate that discontinuities of the magnetic field do indeed occur (Dreher, Ruban, & Grauer 2005). However, as in all numerical models there remains some lingering doubt on the possibility of mistaking a large gradient for a genuine discontinuity, a doubt enhanced by the fact that hyperdiffusivity has been added to the model to stabilize the numerical scheme, and hyperdiffusivity always makes for smooth solutions. We will prove rigorously that under reasonable physical conditions, singularities of classical solutions indeed occur in a finite time. These will be measured by the size of a certain integral of the gradient of $\mathbf{B}$.

Let us finally mention that for phenomena whose frequency lies between the ion and electron gyrofrequencies, and length scales between the ion and electron inertial lengths, one may consider the ions as stationary and the only flow is provided by the flux of electrons. This is the so-called electron MHD model (Gordeev, Kingsep & Rudakov 1994), and is the object of active research, in particular, concerning its turbulence properties (see, e.g. Cho & Lazarian 2005; Wareing & Hollerbach 2009).

The relevant equation in this case is (1.2), taking $\mathbf{v}=\mathbf{0}$. Obviously, our results apply also to this case, although a simpler argument would work in this system.

2. Setting of the problem

Since we will study axisymmetric solutions of (1.1)–(1.3), we choose an appropriate domain and boundary conditions. Ω will be a cylinder with the radius vector $r \in [0, R)$, the height $z \in (-h, h)$, and the azimuthal angle $\phi \in [0, 2\pi]$. We will take the same special form of solutions as in (Dreher *et al.* 2005): the magnetic field $\mathbf{B}$ will be purely toroidal,

$$\mathbf{B} = -r\beta(z, r, t)\mathbf{e}_\phi, \tag{2.1}$$

while the velocity will be poloidal:

$$\mathbf{v} = U\mathbf{e}_z + V\mathbf{e}_r. \tag{2.2}$$

A combination of the vorticity equation derived from (1.1) and the induction one (1.2) yields

$$\frac{\partial}{\partial t}\left(\nabla \times \mathbf{v} + d_i^{-1}\mathbf{B}\right) = \nabla \times \left(\mathbf{v} \times \left(\nabla \times \mathbf{v} + d_i^{-1}\mathbf{B}\right)\right), \tag{2.3}$$

which prompts us to write this transported magnitude as

$$\nabla \times \mathbf{v} + d_i^{-1}\mathbf{B} = r\alpha(z, r, t)\mathbf{e}_\phi. \tag{2.4}$$

Recovering $\nabla \times \mathbf{v}$ from (2.4), we find

$$\frac{\partial V}{\partial z} - \frac{\partial U}{\partial r} = r\left(\alpha + d_i^{-1}\beta\right). \tag{2.5}$$

$\mathbf{v}$ satisfies the incompressibility condition

$$\frac{\partial}{\partial r}(rV) + \frac{\partial}{\partial z}(rU) = 0. \tag{2.6}$$

Since $r\alpha\mathbf{e}_\phi$ is transported by $\mathbf{v}$,

$$\frac{\partial \alpha}{\partial t} + U\frac{\partial \alpha}{\partial z} + V\frac{\partial \alpha}{\partial r} = 0. \tag{2.7}$$

Finally, the induction equation may be written (Dreher *et al.* 2005) as

$$\frac{\partial \beta}{\partial t} + U\frac{\partial \beta}{\partial z} + V\frac{\partial \beta}{\partial r} + 2d_i\beta\frac{\partial \beta}{\partial z} = 0. \tag{2.8}$$

Equations (2.5)–(2.8) form the reduced system. Let us consider now the boundary conditions. If we consider Ω as a domain closed by the flow, we must avoid normal flow in the boundary: $\mathbf{v}\cdot\mathbf{n}=0$ at $\partial\Omega$. In the section $D=(-h, h) \times [0, R]$, this means

$$\begin{aligned} U(\pm h, r, t) &= 0, \quad r \in [0, R], \\ V(z, R, t) &= 0, \quad z \in [-h, h], \\ V(z, 0, t) &= 0, \quad z \in [-h, h]. \end{aligned} \tag{2.9}$$

The last condition follows from the axisymmetry of the flow. As for the magnetic field, it must vanish in the upper and lower lids:

$$\beta(\pm h, r, t) = 0. \tag{2.10}$$

We may also consider that both lids of the cylinder are connected in a torus. In this periodic case, we assume U, V periodic in z with period $2h$, and $\mathbf{v}\cdot\mathbf{n}=0$ at the lateral part of the cylinder:

$$V(z,0,t) = V(z,R,t) = 0, \quad z \in [-h,h]. \tag{2.11}$$

In this case we also need to impose that U has zero mean, which translates into

$$\int_D U(z,r,t) r \, \mathrm{d}z \, \mathrm{d}r = 0. \tag{2.12}$$

As for the magnetic field, both β and β_z must also be periodic in z.

Let us consider (2.7) and (2.8) first. α is constant along the streamlines of the flow $\mathbf{v}$, whereas β is constant along the streamlines of $\mathbf{v} + 2d_i\beta\mathbf{e}_z$, none of which leave the domain Ω (in the periodic case we may think that the upper and lower lids are connected). Therefore if we set initial conditions so that α and β are uniformly bounded at the instant $t=0$, the same happens for all time. We will therefore assume

$$\|\alpha\|_\infty + \|\beta\|_\infty \leqslant M, \tag{2.13}$$

for all time. Since Ω has finite measure, this implies an analogous bound for all the L^p norms of α and β.

If we consider the subspace E of the Sobolev space $W^{1,p}(\Omega)$, $1 < p < \infty$, formed by the solenoidal functions satisfying either the zero normal velocity or the periodic boundary conditions described above, the mapping

$$\left.\begin{array}{l} E \rightarrow L^p(\Omega)^3 \\ \mathbf{v} \rightarrow \nabla \times \mathbf{v} \end{array}\right\} \tag{2.14}$$

is bijective and continuous, and takes the subspace of axisymmetric functions into itself. This is a particular case of the Helmholtz–Weyl decomposition, and it boils down to classical theorems on the regularity of solutions of elliptic equations (see, e.g. Simader & Sohr 1992). For instance, if we take a flux function for the velocity

$$U = \frac{1}{r}\frac{\partial\psi}{\partial r}, \quad V = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \tag{2.15}$$

for the zero normal velocity boundary condition we may take $\psi=0$ at ∂D, and for the periodic one ψ equal to zero at $r=0$ and $r=R$, and periodic in z. Equation (2.5) turns out to be an elliptic equation in ψ, for which we have well known theorems of regularity which prove our previous statement. Notice that this fails for $p=\infty$. Therefore there exist constants C_p such that

$$\|\mathbf{v}\|_{W^{1,p}} \leqslant C_p \|\nabla \times \mathbf{v}\|_p \tag{2.16}$$

for all functions in E. In our case, (2.5) implies that the $W^{1,p}$ norm of U and V are bounded by a constant times the L^p norm of $r(\alpha + d_i\beta)$. This, together with (2.13), implies that the partial derivatives of U and V are bounded in L^p norm by a constant $C_pM = M_p$.

From now on we will denote partial derivatives by subindices. We therefore have

$$\|U_z\|_p, \ \|U_r\|_p, \ \|V_z\|_p, \ \|V_r\|_p \leqslant M_p \tag{2.17}$$

for all time. Notice that with our assumptions, both $U\beta_z$ and $\beta\beta_z^3$ are periodic in z, in the first case because both are zero at $z=\pm h$.

3. The main inequalities

Differentiating (2.8) with respect to z, we obtain (Dreher *et al.* 2005)

$$\frac{\partial \beta_z}{\partial t} + U_z\beta_z + U\beta_{zz} + V_z\beta_r + V\beta_{rz} + 2d_i\beta_z^2 + 2d_i\beta\beta_{zz} = 0. \tag{3.1}$$

Therefore the function $\gamma = -\beta$ satisfies

$$\frac{\partial \gamma_z}{\partial t} + U_z\gamma_z + U\gamma_{zz} + V_z\gamma_r + V\gamma_{rz} - 2d_i\gamma_z^2 - 2d_i\gamma\gamma_{zz} = 0. \tag{3.2}$$

Multiplying this by γ_z^2, we get

$$\frac{1}{3}\frac{\partial \gamma_z^3}{\partial t} + U_z\gamma_z^3 + U\gamma_z^2\gamma_{zz} + V_z\gamma_z^2\gamma_r + V\gamma_z^2\gamma_{rz} - 2d_i\gamma_z^4 - 2d_i\gamma\gamma_z^2\gamma_{zz} = 0. \tag{3.3}$$

Let us integrate each of these terms in Ω:

$$\frac{1}{3}\int_\Omega \frac{\partial \gamma_z^3}{\partial t}\,\mathrm{d}V = \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \gamma_z^3\,\mathrm{d}V. \tag{3.4}$$

By Hölder's inequality,

$$\left|\int_\Omega U_z\gamma_z^3\,\mathrm{d}V\right| \leqslant \|U_z\|_4\|\gamma_z\|_4^3. \tag{3.5}$$

Integration by parts yields

$$\int_\Omega U\gamma_z^2\gamma_{zz}\,\mathrm{d}V = \frac{1}{3}\int_\Omega U(\gamma_z^3)_z\,\mathrm{d}V = \frac{2\pi}{3}\int_0^R r\left[U\gamma_z^3\right]_{z=-h}^{z=h}\mathrm{d}r - \frac{1}{3}\int_\Omega U_z\gamma_z^3\,\mathrm{d}V. \tag{3.6}$$

The first integral vanishes because of our hypotheses on the periodicity of U, β and β_z (which translate to γ). The second one is identical to the one in (3.5) and may be bounded in the same way.

Again by Hölder's inequality,

$$\left|\int_\Omega V_z\gamma_z^2\gamma_r\,\mathrm{d}V\right| \leqslant \|V_z\|_6\|\gamma_r\|_3\|\gamma_z\|_4^2. \tag{3.7}$$

For the next term we use again integration by parts:

$$\begin{aligned}\int_\Omega V\gamma_z^2\gamma_{zr}\,\mathrm{d}V &= \frac{1}{3}\int_\Omega V(\gamma_z^3)_r\,\mathrm{d}V\\ &= \frac{2\pi}{3}\int_{-h}^{h}\left[rV\gamma_z^3\right]_{r=0}^{r=R}\mathrm{d}z - \frac{2\pi}{3}\int_D (V + rV_r)\gamma_z^3\,\mathrm{d}z\,\mathrm{d}r. \end{aligned}\tag{3.8}$$

The first integral vanishes because $V(z,0) = V(z,R) = 0$. The second one has two components:

$$\frac{2\pi}{3}\int_D rV_r\gamma_z^3\,\mathrm{d}z\,\mathrm{d}r = \frac{1}{3}\int_\Omega V_r\gamma_z^3\,\mathrm{d}V, \tag{3.9}$$

which analogously to the integral in (3.5), may be bounded by

$$\frac{1}{3}\|V_r\|_4\|\gamma_z\|_4^3. \tag{3.10}$$

As for

$$\frac{2\pi}{3}\int_D V\gamma_z^3\,\mathrm{d}z\,\mathrm{d}r = \frac{1}{3}\int_\Omega \frac{1}{r}V\gamma_z^3\,\mathrm{d}V, \tag{3.11}$$

it is apparently singular at $r=0$. In fact, classical results on the behaviour of V near the axis guarantee that V/r is bounded there, but we can obtain the bound we need by elementary means. Obviously the integral in z plays no role at all, and it is enough to study the integral with respect to r. We have

$$\left|\int_0^R V\gamma_z^3\,\mathrm{d}r\right| \leqslant \left(\int_0^R \gamma_z^4 r\,\mathrm{d}r\right)^{3/4}\left(\int_0^R \frac{1}{r}V^4\,\mathrm{d}r\right)^{1/4} = \|\gamma_z\|_4^3\left(\int_0^R \frac{1}{r}V^4\,\mathrm{d}r\right)^{1/4}. \quad (3.12)$$

Since

$$V(r) = \int_0^r s^{-1/4}s^{1/4}V_r(s)\,\mathrm{d}s, \quad (3.13)$$

we have

$$|V(r)| \leqslant \left(\int_0^r s^{-1/3}\,\mathrm{d}s\right)^{3/4}\left(\int_0^r sV_r(s)^4\,\mathrm{d}s\right)^{1/4}$$
$$\leqslant \left(\frac{3}{2}r^{2/3}\right)^{3/4}\|V_r\|_4 = \left(\frac{3}{2}\right)^{3/4} r^{1/2}\|V_r\|_4. \quad (3.14)$$

Hence

$$\left(\int_0^R \frac{1}{r}V^4\,\mathrm{d}r\right)^{1/4} \leqslant \|V_r\|_4\left(\int_0^R\left(\frac{3}{2}\right)^{3/4} r^{-1/2}\,\mathrm{d}r\right)^{1/4} = 3^{3/8}2^{1/8}R^{1/8}\|V_r\|_4. \quad (3.15)$$

This, together with (3.12), yields

$$\left|\int_\Omega V\gamma_z^2\gamma_{zr}\,\mathrm{d}V\right| \leqslant C\|V_r\|_4\|\gamma_z\|_4^3 \quad (3.16)$$

for a constant C. As for the nonlinear terms,

$$-2d_i\int_\Omega \gamma_z^4\,\mathrm{d}V = -2d_i\|\gamma_z\|_4^4, \quad (3.17)$$

and

$$-2d_i\int_\Omega \gamma\gamma_z^2\gamma_{zz}\,\mathrm{d}V = -\frac{2d_i}{3}\int_\Omega \gamma\left(\gamma_z^3\right)_z\mathrm{d}V$$
$$= -\frac{4\pi d_i}{3}\int_0^R r\left[\gamma\gamma_z^3\right]_{z=-h}^{z=h}\mathrm{d}r + \frac{2d_i}{3}\int_\Omega \gamma_z^4\,\mathrm{d}V. \quad (3.18)$$

Again the first integral vanishes. Thus the sum of (3.17) and (3.18) yields

$$-2d_i\int_\Omega \gamma_z^4 + \gamma\gamma_z^2\gamma_{zz}\,\mathrm{d}V = -\frac{4d_i}{3}\|\gamma_z\|_4^4. \quad (3.19)$$

Let us recall that all the partial derivatives of U and V are bounded in L^4 and L^6 norm. Thus from integration of (3.3) we find that there exists another constant, again denoted by C, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \gamma_z^3\,\mathrm{d}V \geqslant \frac{4d_i}{3}\|\gamma_z\|_4^4 - C\|\gamma_z\|_4^3 - C\|\gamma_r\|_3\|\gamma_z\|_4^2, \quad (3.20)$$

the last term proceeding from (3.7).

There exist two possibilities. The first one is that $\|\gamma_r\|_3$ tends to infinity at a finite time T_*, in whose case obviously $\|\nabla\mathbf{B}\|_3$ undergoes a blow-up at a finite time; or $\|\gamma_r\|_3$ may be finite for all time (which does not mean to be uniformly bounded). Nonetheless it is bounded in every finite interval $[0, T]$. Thus there exists a constant L_T such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \gamma_z^3\,\mathrm{d}V \geqslant \frac{4d_i}{3}\|\gamma_z\|_4^4 - C\|\gamma_z\|_4^3 - L_T\|\gamma_z\|_4^2, \tag{3.21}$$

for all $t \in [0, T]$. Since the polynomial

$$p(x) = \frac{d_i}{3}x^4 - Cx^3 - L_T x^2, \tag{3.22}$$

tends to ∞ when $x \to \infty$, it has a finite minimum $-\lambda_T$. Then

$$\frac{4d_i}{3}x^4 - Cx^3 - L_T x^2 \geqslant d_i x^4 - \lambda_T, \tag{3.23}$$

for all x, and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \gamma_z^3\,\mathrm{d}V \geqslant d_i\|\gamma_z\|_4^4 - \lambda_T. \tag{3.24}$$

Notice that λ_T does not depend on T if $\|\gamma_r\|_3$ is uniformly bounded. Hölder's inequality implies

$$\|\gamma_z\|_4^4 \geqslant (4\pi h R)^{-1/3}\|\gamma_z\|_3^4. \tag{3.25}$$

Hence, calling $\mu = d_i(4\pi h R)^{-1/3}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \gamma_z^3\,\mathrm{d}V \geqslant \mu\|\gamma_z\|_3^{4/3} - \lambda_T. \tag{3.26}$$

Since obviously

$$|x(t)| = \left|\int_\Omega \gamma_z^3\,\mathrm{d}V\right| \leqslant \|\gamma_z\|_3^3, \tag{3.27}$$

we obtain

$$x'(t) \geqslant \mu|x(t)|^{4/3} - \lambda_T, \tag{3.28}$$

for all $t \in [0, T]$. Hence, if $x(0) > 0$, $\mu x(0)^{4/3} > \lambda_T$, x is always increasing and in particular always positive. In this case, $x(t) \to \infty$ when $t \to T_*$, where

$$T_* = \int_{x(0)}^{\infty} \frac{\mathrm{d}x}{\mu x^{4/3} - \lambda_T} < \infty, \tag{3.29}$$

provided $T_* \leqslant T$. Notice that since the integral of this function is finite, this happens sooner or later if λ_T is uniformly bounded for all T. If this does not occur, nevertheless blow-up occurs for large enough $x(0)$ to have

$$\int_{x(0)}^{\infty} \frac{\mathrm{d}x}{\mu x^{4/3} - \lambda_T} \leqslant T. \tag{3.30}$$

Again the existence of such $x(0)$ is guaranteed. Therefore we can always choose smooth initial conditions such that the function x and a fortiori $\|\gamma_z\|_3$ tends to infinity, unless $\|\gamma_r\|_3$ tends to infinity first. Anyway a finite time blow-up of $\|\nabla\mathbf{B}\|_3$ is guaranteed for particular smooth initial conditions; if $\|\mathbf{B}_r\|_3$ is bounded in time, for all initial conditions larger than a fixed constant. Notice that since in our case $\|\nabla\mathbf{B}\|_3$ and $\|\mathbf{J}\|_3$ are equivalent norms, what we find is a blow-up of the current density in the L^3 norm.

4. Conclusions

There is considerable numerical evidence of the formation of shocks of the magnetic field in Hall MHD, but a rigorous proof of their existence was lacking. If we interpret this singularity formation as a blow-up of the gradient of the magnetic field in L^3 integral norm, we may choose a certain configuration of the velocity and the magnetic field in an incompressible plasma that will certainly lead to this blow-up. Specifically, in this geometry all the magnitudes are axisymmetric, the velocity is assumed poloidal and the magnetic field toroidal. In this case, if we assume that the magnetic field and the vorticity are uniformly bounded at the initial instant, the same thing happens for all time. By contrast, the vertical derivative of the magnetic field satisfies a nonlinear evolution equation which may be integrated in the domain for appropriate boundary conditions. The new scalar differential inequality guarantees a finite time blow-up for large enough initial conditions. This means that $\|\nabla\mathbf{B}\|_3$ (or equivalently $\|\mathbf{J}\|_3$) tends to blow up at a finite time. This particular geometry of the flow and the magnetic field is chosen to enhance the effect of the Hall term, that is, the transport of magnetic field by the electron flow, but it is likely that blow-ups occur for many other topologies.

REFERENCES

BISKAMP, D. 2000 *Magnetic Reconnection in Plasmas*. Cambridge University press.

CHEN, F. F. 1983 *Introduction to Plasma Physics and Controlled Fusion*. Plenum Press.

CHO, J. & LAZARIAN, A. 2004 The anisotropy of electron magnetohydrodynamic turbulence. *Astrophys. J. Lett.* **615**, L41–L44.

DENG, J., HOU, T. Y. & YU, X. 2006 Improved geometric conditions for non-blowup of the three-dimensional incompressible Euler equation. *Commun. Part. Differ. Equ.* **31**, 293–306.

DREHER, J., RUBAN, V. & GRAUER, R. 2005 Axisymmetric flows in Hall MHD: a tendency towards finite-time singularity formation. *Phys. Scr.* **72**, 451–455.

GIBBON, J. D., MOORE, D. R. & STUART, J. T. 2003 Exact, infinite energy, blow-up solutions of the three-dimensional Euler equations. *Nonlinearity* **16**, 1823–1831.

GORDEEV, A. V., KINGSEP, A. S. & RUDAKOV, L. I. 1994 Electron magnetohydrodynamics. *Phys. Rep.* **243** (5), 215–315

KERR, R. M. 1993 Evidence for a singularity of the three-dimensional incompressible Euler equations. *Phys. Fluids* A **5**, 1725–1746.

LI, T. & WANG, D. 2006 Blow up phenomena of solutions to the Euler equations for compressible fluid flow. *J. Differ. Equ.* **221**, 91–101.

PANDEY, B. P. & WARDLE, M. 2008 Hall magnetohydrodynamics of partially ionized plasmas. *MNRAS* **385** (4), 2269–2278.

PELTZ, R. B. 2001 Symmetry and the hydrodynamic blow-up problem. *J. Fluid Mech.* **444**, 299–320.

SCHEKOCHIHIN, A. A., COWLEY, S. C., DORLAND, W., HAMMETT, G. W., HOWES, G. G., QUATAERT, E. & TATSUNO, T. 2009 Astrophysical gyrokinetics: kinetic and fluid turbulent cascades in magnetized weakly collisional plasmas. *Astrophys. J. Suppl. Ser.* **182** (1), 310–377.

SHAY, M. A., DRAKE, J. F., ROGERS, B. N. & DENTON, R. E. 2001 Alfvénic collisionless magnetic reconnection and the Hall term. *J. Geophys. Res.* **106**, 3759–3772.

SIMADER, C. G. & SOHR, H. 1992 A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains. In *Mathematical Problems Relating* *to the Navier–Stokes Equations* (ed. G. P. Galdi), Ser. Adv. Math. Appl. Sci., pp. 1–35. World Scientic.

SONNERUP, B. U. O. 1970 Magnetic field reconnection in a highly conducting incompressible fluid. *J. Plasma Phys.* **4**, 161–174.

WAREING, C. J. & HOLLERBACH, R. 2009 Forward and inverse cascades in decaying two-dimensional electron magnetohydrodynamic turbulence. *Phys. Plasmas* **16**, 042307.

YEH, T. & AXFORD, W. I. 1970 On the reconnection of magnetic field lines in conducting fluids. *J. Plasma Phys.* **4**, 207–229.